Inverting the Cantor-Bendixson Derivative

Jason Reed

November 13, 2000

Abstract

In a suitable topological space X, the Cantor-Bendixson derivative D_X (which takes a set to the set of its accumulation points) has a pseudo-inverse I such that $D_X IC = C$ for any closed C. In particular, when X is a metric space, we show there exists an I with this property which also preserves arbitrary unions.

1 Introduction

The notion of the the set of accumulation points of a subset A of $\mathbb{R}^n$ (also called the *derived set* of A, the *derivative* of A, or the *Cantor-Bendixson derivative* of A due to its central role in the Cantor-Bendixson theorem) was introduced by Cantor, and was fundamental in motivating his theory of transfinite cardinals. (see [Hal84]) More generally, the set of accumulation points of a subset A of a topological space X is defined by

$$D_X A := \{x \in X \mid x \in \overline{A - \{x\}}\} \tag{1}$$

We will drop the subscript on D when the space is clear from context. This notation is not standard, but like the more common notation of A' for the derived set of A, it suggests that the operation is like differentiation, and moreover is considered as a map $D_X : \mathcal{P}X \to \mathcal{P}X$. Indeed we easily have

Fact 1.1 $D(A \cup B) = DA \cup DB$

which appears analogous to the additivity

$$\frac{d}{dx}(f + g) = \frac{d}{dx}f + \frac{d}{dx}g$$

of the usual differential operator on real-valued functions. Motivated by this terminology, we seek to determine when there exists an *antiderivative* IA of a given set A, such that $DIA = A$. Other facts which will prove useful later (but which have no direct analytic counterpart) are

Fact 1.2 $\overline{A} = A \cup DA$

and

Fact 1.3 *If* $A \subseteq B$*, then* $DA \subseteq DB$*.*

which follow easily from (1).

2 $T_{1/2}$ spaces

It happens that in 'most' spaces, the range of D consists of only closed sets, so that we can only hope to find an antiderivative IC for closed C. Indeed, [KM76] adds T_1 (which by Proposition 2.1 implies that this is the case) to its definition of a topological space just to define accumulation points. We will indeed want that $\operatorname{rng} D_X \subseteq \mathcal{C}X$ (where $\mathcal{C}X$ is the collection of closed subsets of X) in all the spaces we work with, but it turns out that this condition isn't quite as strong as T_1. Hence we define

Definition A space X is said to be $T_{1/2}$ if $D_X A$ is closed for any $A \subseteq X$.

because of the following pair of results:

Proposition 2.1 *All T_1 spaces are $T_{1/2}$, but not conversely.*

Proof Let a subset $A \subseteq X$ be given. We want to show DA is closed. So let $x \in \overline{DA}$ and a neighborhood U of x be given. Since x is in the closure of DA, we can choose $y \in U \cap DA$. If $x = y$, then $x \in DA$ and we are done. Otherwise, we use the T_1 axiom to find a neighborhood V of y not containing x, and observe that $V \cap U$ is a also neighborhood of y, and since $y \in DA$, we can choose $z \in V \cap U$ such that also $v \in A$, and by choice of V we know $z \neq x$. That is, for every neighborhood U of x there exists $z \in U$ such that also $z \neq x$ and $z \in A$. Hence $x \in DA$ so DA is closed. To see that not all $T_{1/2}$ spaces are T_1, consider the space $X := \{*, \star\}$ where the open sets are $\emptyset$, $\{*\}$, and X. (See Figure 1) Then $D\{*, \star\} = D\{*\} = \{\star\}$, which is closed, and $D\{\star\} = D\emptyset = \emptyset$,

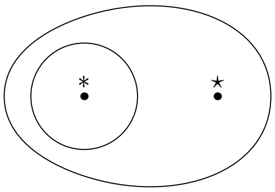

Figure 1: $T_{1/2}$ yet not T_1 space

so X is $T_{1/2}$. But every neighborhood of $\star$ contains $*$, so X is not T_1. ■

Proposition 2.2 *All $T_{1/2}$ spaces are T_0, but not conversely.*

Proof Suppose X is $T_{1/2}$. Let x, y with $x \neq y$ be given. Suppose every neighborhood of y contains x. Then $y \in D\{x\}$ and also $x \notin D\{x\}$, since no neighborhood of x contains any point of $\{x\}$ distinct from x. Since X is $T_{1/2}$, we know $D\{x\}$ is closed and so $X - D\{x\}$ (which contains x but not y) must be open. So if every neighborhood of y contains x, then not every neighborhood of x contains y, hence X is T_0. To see that not all T_0 spaces are $T_{1/2}$, consider the space $X := \{0, 1, 2, 3, 4, \ldots \infty\}$ where the open sets are $\emptyset$, X, and

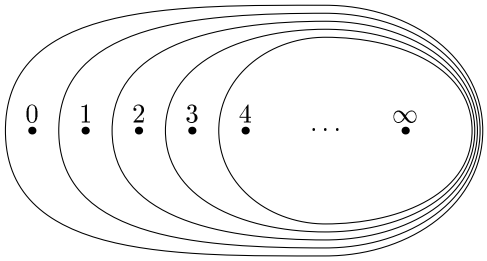

Figure 2: T_0 yet not $T_{1/2}$ space

$G_m := \{n \mid n \geq m\} \cup \{\infty\}$ for $m \in \mathbb{N}$. (See Figure 2) Then $D\{\infty\} = \mathbb{N}$, yet $\mathbb{N}$ is not closed, so X is not $T_{1/2}$. But for any $m, m' \in \mathbb{N}$, (assuming without loss of generality that $m' > m$) $G_{m'}$ contains m' but not m. And for any $m \in \mathbb{N}$, we have that G_{m+1} contains ∞ but not m. So X is T_0. ■

Another convenient property (which will be used in Proposition 3.2) of $T_{1/2}$ spaces is the following:

Proposition 2.3 *If X is a $T_{1/2}$ space, then $D\overline{A} = DA$.*

Proof $D\overline{A} = D(A \cup DA)$ by Fact 1.2, and $D(A \cup DA) = DA \cup DDA$ by Fact 1.1. Using Fact 1.2 again, $DA \cup DDA = \overline{DA}$. But X is $T_{1/2}$, so $\overline{DA} = DA$. ■

3 Integrals

3.1 Defining integrals

If X is a $T_{1/2}$ space, then we know $\operatorname{rng} D_X \subseteq \mathcal{C}X$. Optimistically consider spaces in which $\operatorname{rng} D_X = \mathcal{C}X$. In this case, we can find a function $I \colon \mathcal{C}X \to \mathcal{P}X$ such that $DIC = C$ for every $C \in \mathcal{C}X$. We call a function with this property an *integral* on X. Continuing our optimism, we also consider integrals which have the same properties that we expect D to have;

Definition An integral I on X is *finitely additive* if $I(A \cup B) = IA \cup IB$ for any $A, B \in \mathcal{C}X$.

Definition An integral on I on X is *closed* if $\operatorname{rng} I \subseteq \mathcal{C}X$.

However, the first of these conditions is somewhat stronger than it appears, for if we say

Definition An integral I on X is *additive* if

$$\bigcup_{j \in J} I(C_j) = I\left(\bigcup_{j \in J} C_j\right)$$

for any family $(C_j)_{j\in J}$ of closed sets of X.

then we have

Proposition 3.1 *If there exists a finitely additive integral on X, then there exists an additive integral on X.*

Proof Let I be a finitely additive integral. Define I_1 by

$$I_1 C := \bigcup_{c\in C} I\{c\}$$

I_1 is clearly additive by definition. We need to show that $DI_1C = C$ for any closed C.

- $C \subseteq DI_1C$: Let $c \in C$ be given. $I\{c\} \subseteq I_1C$, so $\{c\} = DI\{c\} \subseteq DI_1C$ (by Fact 1.3) or equivalently $c \in DI_1C$.
- $DI_1C \subseteq C$: For all $c \in C$ we know $C \cup \{c\} = C$, hence $IC \cup I\{c\} = I(C \cup \{c\}) = IC$ and so we see $I\{c\} \subseteq IC$. Thus $I_1C = \bigcup_{c\in C} I\{c\} \subseteq IC$, and so $DI_1C \subseteq DIC = C$ again by Fact 1.3. ■

Furthermore, the second condition is not particularly restrictive at all, if we are concerned with existence:

Proposition 3.2 *Let X be a $T_{1/2}$ space. Then*

i) If there exists an integral on X, then there exists a closed integral on X.

ii) there exists an additive integral on X, then there exists a closed additive integral on X.

Proof i) Let I be an integral on X. Define $I^*(C) := \overline{I(C)}$. Then I^* is an integral, for (using Proposition 2.3) $D_X(I^*(C)) = D_X(\overline{I(C)}) = D_X(I(C)) = C$.

ii) Define I^* as above. We additionally assume that I is additive, and additionally need to show that I^* is additive. But this is easily seen to be the case:

$$\begin{aligned} I^*\left(\bigcup_{j\in J} C_j\right) &= \overline{I\left(\bigcup_{j\in J} C_j\right)} \\ &= \left(I\left(\bigcup_{j\in J} C_j\right)\right) \cup D\left(I\left(\bigcup_{j\in J} C_j\right)\right) \\ &= \left(I\left(\bigcup_{j\in J} C_j\right)\right) \cup \left(\bigcup_{j\in J} C_j\right) \end{aligned}$$

$$
\begin{aligned}
&= \left(\bigcup_{j\in J} I(C_j)\right) \cup \left(\bigcup_{j\in J} C_j\right) \\
&= \bigcup_{j\in J} I(C_j) \cup C_j \\
&= \bigcup_{j\in J} I(C_j) \cup D(I(C_j)) \\
&= \bigcup_{j\in J} \overline{I(C_j)} \\
&= \bigcup_{j\in J} I^*(C_j)
\end{aligned}
$$

■

3.2 Finding integrals

Ignoring additivity for now, how would we go about trying to find an antiderivative for some closed set C? Since C is closed, we know $DC \subseteq C$, so we can start with C and try to add enough points to hit all of $C - DC$, the isolated points of C. Consider a scenario resembling Figure 3, where we can find disjoint open

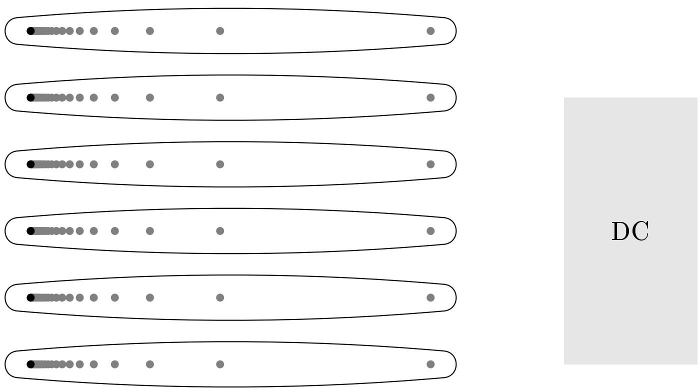

Figure 3: Motivation for discernibility

sets around all the isolated points, and for each isolated point x find a sequence converging to exactly x; perhaps the accumulation points of all these sequences would be just the isolated points of C, and the accumulation points of C would be the rest of C. We will want to assume first countability and Hausdorff to provide sequences, but we need to be sure that the sequences don't 'interfere' and produce unwanted points of accumulation. Hence we make the following definition:

Definition A space X is *discernible* iff X is Hausdorff and, for any set $P \subseteq X$

such that $P \cap DP = \emptyset$, (that is, every $p \in P$ is an isolated point of P) there exists a family $(U_p)_{p \in P}$ of disjoint open sets such that

i) $p \in U_p$ for all $p \in P$.

ii) If $(V_p)_{p \in P}$ is a family of finite subsets of X such that $V_p \subseteq U_p$ for all $p \in P$, then $\overline{\bigcup_{p \in P} V_p} \subseteq \overline{P}$.

First of all, merely being able to find disjoint open sets around isolated points together with first countability together imply regularity;

Proposition 3.3 *Let X be a space. If X is discernible and first countable, then X is regular.*

Proof Let $C \in \mathcal{C}X$ and $x \in X$ be given. Suppose that for any open $V \supseteq C$ and open $U \ni x$, then $V \cap U \neq \emptyset$. Choose a local basis $(B_n)_{n \in \mathbb{N}}$ of x, and set $U_m = \bigcap_{n \leq m} B_n$. Observe that $(U_m)_{m \in \mathbb{N}}$ is also a local basis of x, but also $U_m \subseteq U_{m'}$ whenever $m \geq m'$. Now for every $n \in \mathbb{N}$, we can by assumption choose c_n such that every neighborhood of c_n meets U_n, for if were the case that every $c \in C$ had a neighborhood W_c such that W_c didn't meet U_n, then $\bigcup_{c \in C} W_c$ would be an open set containing C which doesn't meet U_n, contradicting our choice of C, x. So consider the family $(c_n)_{n \in \mathbb{N}}$; set $C' := \{c_n \mid n \in \mathbb{N}\}$. We claim that c_m is an isolated point of C' for every $m \in \mathbb{N}$. Since X is Hausdorff, we can choose disjoint open neighborhoods U, W of x, c_m respectively. But $(U_n)_{n \in \mathbb{N}}$ is a local basis of x, so we can choose p such that $U_p \subseteq U$ and clearly then U_p and W are disjoint. From this we can deduce that $W \not\ni c_q$ for any $q \geq p$, since if $c_q \in W$, then (by choice of $(c_n)_{n \in \mathbb{N}}$) $W \cap U_q \neq \emptyset$ and since $U_q \subseteq U_p$, we would have $W \cap U_p \neq \emptyset$, a contradiction. So $W \cap C'$ is finite, being contained in $\{c_0, c_1, \ldots, c_{p-1}\}$. We can then repeatedly apply the Hausdorff axiom to produce a W' such that $W' \cap C' = \{c_m\}$, and therefore the claim holds. Now $C' \cup \{x\}$ is countable, yet we assumed that we cannot separate x from all of C' by disjoint open sets. By discernibility we know that x must not be isolated from all of C', since we have just shown that every point of C' is isolated. That is, $x \in \overline{C}' \subseteq \overline{C} = C$. Hence if $x \notin C$, then we **can** separate x from C by disjoint open sets and X is regular. ■

Using this, we can show the desired existence result for integrals:

Proposition 3.4 *Let X be a space. If X is first countable, discernible, and has no isolated points, then there exists an integral I on X.*

Proof Let $C \in \mathcal{C}(X)$ be given. Set $P := C - DC$. We see that $P \subseteq C$ so $DP \subseteq DC$ by Fact 1.3, hence $(C - DC) \cap P = \emptyset$. So by discernibility we can choose suitable $(U_p)_{p \in P}$. Find a sequence $(a_{p,n})_{n \in \mathbb{N}}$ in $X - \{p\}$ converging to p for each $p \in P$, such that there are only finitely many points of the sequence outside of any neighborhood of p. (This is easily accomplished by a recursive construction using first countability, Hausdorff, and the lack of isolated points) Now define

$$IC := C \cup \bigcup_{p \in P} U_p \cap \{a_{p,n} \mid n \in \mathbb{N}\} \tag{2}$$

We need to show that $DIC = C$.

- $C \subseteq DIC$: Let $c \in C$ be given. If $c \in DC$, then $c \in DIC$ because $C \subseteq IC$. Otherwise, $c \in C - DC = P$. By assumption $(a_{c,n})_{n\in\mathbb{N}}$ converges to c, so for any neighborhood V of c we can find $n \in \mathbb{N}$ such that $a_{c,n} \in U_p \cap V \subseteq V$. In this case $a_{c,n} \in IC$, and recall that we also have that $(a_{c,n})_{n\in\mathbb{N}}$ is a sequence in $X - \{c\}$, so $a_{c,n} \neq c$, and $c \in DIC$.
- $DIC \subseteq C$: Let $d \in DIC$ be given. By (2) and Fact 1.1, either $d \in DC$ or $d \in D\left(\bigcup_{p\in P} U_p \cap \{a_{p,n} \mid n \in \mathbb{N}\}\right)$. If $d \in DC$, then we are done, since $DC \subseteq DC \cup C = \overline{C} = C$. Otherwise, suppose towards a contradiction that $d \notin \overline{P}$. By Proposition 3.3 we can find disjoint neighborhoods V and W of $\overline{P}$ and d, respectively. By first countability we can then choose a sequence $(b_m)_{m\in\mathbb{N}}$ converging to d such that for each m, there exists $p \in P$ such that $b_m \in W \cap U_p \cap \{a_{p,n} \mid n \in \mathbb{N}\}$. Because V, W are disjoint, this sequence stays outside of a neighborhood of each $p \in P$, and by choice of $(a_{p,n})_{n\in\mathbb{N}}$, the sequence only meets each U_p finitely often; by discernibility, the closure of the range of the sequence is contained in $\overline{P}$, and so specifically $d \in \overline{P}$, a contradiction, forcing us to conclude that $d \in \overline{P} \subseteq \overline{C} = C$. ■

3.3 Existence of Additive Integrals

The optimism in §3.1 has not yet been completely justified; it is not at all clear that the construction of Proposition 3.4 can be modified to produce an additive integral, since we make choices about which sequence to assign to each isolated point based on the points around it, not to mention deciding whether to assign a sequence at all based on whether the point is isolated or not. Yet we can establish a somewhat surprising positive result: a closed additive integral exists in any metric space without isolated points. We first prove a lemma which establishes the existence of a sequence of 'layers' of points in a metric space. Each of these layers has no accumulation points, but together they eventually become sufficiently dense so as to permit definition of the desired integral based on including points from these layers 'near enough' to the closed set we want to find a preimage of under D.

Lemma 3.5 *Let X be a metric space with metric δ. Assume X has no isolated points. Then there exists a sequence $(E_m)_{m\in\mathbb{N}}$ of disjoint subsets of X such that $D(E_m) = \emptyset$ and $\overline{\bigcup_{n\geq m} E_n} = X$ for all $m \in \mathbb{N}$.*

Proof We define $\mathcal{E}_m \subseteq \mathcal{P}X$ and $E_m \subseteq X$ by recursion on m. At each stage of the recursion, we let

$$\mathcal{E}_m := \{E \subseteq X \setminus \bigcup_{n<m} E_n \mid \neg\exists x, y \in E\colon \delta(x, y) < 1/(m+1)\} \tag{3}$$

and observe that the union of any chain in $\mathcal{E}_m$ ordered by inclusion is again in $\mathcal{E}_m$, hence by Zorn's lemma, we can pick a maximal element $E_m \in \mathcal{E}_m$. First of all, clearly $E_m \cap E_{m'}$ when $m \neq m'$ simply because (assuming $m' > m$ without loss of generality) $E_{m'} \subseteq X \setminus \bigcup_{n<m} E_n \subseteq X \setminus E_m$. We then claim that $D(E) = \emptyset$ for any $m \in \mathbb{N}$ and $E \in \mathcal{E}_m$, so that in particular $D(E_m) = \emptyset$. For let $x \in X$ be given; the open ball $B_{1/(2(m+1))}(x)$ can contain at most one point of E, because if $w, z \in B_{1/(2(m+1))}(x)$ and $w \neq z$, then $\delta(w,x) < 1/(2(m+1))$ and $\delta(z,x) < 1/(2(m+1))$, so by the triangle inequality $\delta(w,z) < 1/(m+1)$, which by (3) means that no element of $\mathcal{E}_m$ contains both w and z. If necessary, we can shrink the ball until it contains no points of E, except possibly x itself. Hence x is not an accumulation point of E, and E can have no accumulation points. It remains to show that $\overline{\bigcup_{n\geq m} E_n}$ is dense in X for every $m \in \mathbb{N}$. Suppose to the contrary that we can find $m \in \mathbb{N}$, $x \in X$ and $\epsilon > 0$ such that

$$\bigcup_{n\geq m} E_n \cap B_\epsilon(x) = \emptyset \tag{4}$$

Then, since $D\left(\bigcup_{n<m} E_n\right) = \emptyset$, we can choose a neighborhood U of x such that

$$U \cap \left(\bigcup_{n<m} E_n\right) \setminus \{x\} = \emptyset \tag{5}$$

Now by assumption X has no isolated points, so that $U \cap B_\epsilon(x)$, being open, contains at least one point other than x, say y. We can fit a ball centered at y with radius $1/(q+1)$ (for some $q \geq m$) entirely inside the open set $U \cap B_\epsilon(x) \setminus \{x\}$. But notice that then (5) implies $B_{1/(q+1)}(y) \cap \bigcup_{n<m} E_n = \emptyset$ because $B_{1/(q+1)}(y) \subseteq U$ and $x \notin B_{1/(q+1)}(y)$. This can be strengthened to $B_{1/(q+1)}(y) \cap \bigcup_{n<q} E_n = \emptyset$ by (4). Using (4) again, we have $\neg\exists x' \in E_q : \delta(x', y) < 1/(q+1)$ since $B_{1/(q+1)}(y) \subseteq B_\epsilon(x)$ and $q \geq m$. So in fact E_q is not maximal, for we have just shown that $E_q \cup \{y\} \in \mathcal{E}_q$. Hence our choice of m, x, ϵ to satisfy (4) was impossible, and $\overline{\bigcup_{n\geq m} E_n} = X$. ∎

Theorem 3.6 *If X be a metric space with metric δ. Assume X has no isolated points. Then there exists a closed additive integral on X.*

Proof Take $(E_m)_{m\in\mathbb{N}}$ as in Lemma 3.5. Define $I_m : \mathcal{C}X \to \mathcal{P}X$ for each $m \in \mathbb{N}$ by

$$I_m(C) := \bigcup_{c\in C} \{e \mid \exists n \geq m : e \in E_n \wedge \delta(e,c) < 1/n\} \tag{6}$$

We claim that

$$\forall m, m' \in \mathbb{N} : D(I_m) = D(I_{m'}) \tag{7}$$

Observe that $I_m \setminus I_{m+1} \subseteq E_m$, and so $D(I_m \setminus I_{m+1}) \subseteq D(E_m) = \emptyset$ by Fact 1.3. But then since $I_{m+1} \subseteq I_m$ we see that $D(I_m) = D(I_{m+1} \cup (I_m \setminus I_{m+1})) = D(I_{m+1}) \cup D(I_m \setminus I_{m+1}) = D(I_{m+1})$ and by induction the claim holds. We now show that I_0 is an integral.

- $D(I_0(C)) \subseteq C$: Let $d \in D(I_0(C))$ and $m \in \mathbb{N}$ be given. We know by (7) that $D(I_0(C)) = D(I_{2m}(C))$, so $d \in D(I_{2m}(C))$, and therefore we can choose $e \in I_{2m}(C)$ such that $\delta(d,e) < 1/(2m)$. But by (6) we can find $c \in C$ such that $\delta(e,c) < 1/(2m)$ and by the triangle inequality $\delta(d,c) < 1/m$. That is, we can get arbitrarily close to d using points of C, hence $d \in \overline{C} = C$ (since C was assumed to be closed) and $D(I_0(C)) \subseteq C$.

- $C \subseteq D(I_0(C))$: Let $c \in C$ and $m \in \mathbb{N}$ be given. We know $\overline{\bigcup_{n \geq m} E_n} = X$, so we can find $n \geq m$ and $e \in E_n$ such that $\delta(e,c) < 1/n \leq 1/m$ and also $e \neq c$. Guaranteeing $e \neq c$ is possible since the E_n are disjoint; c can only show up in once. So for arbitrary m, we can find $e \in I_0(C)$ such that $0 < \delta(e,c) < 1/m$, hence $c \in D(I_0(C))$.

That I_0 is additive is easily seen; given any family $(C_j)_{j \in J}$ of closed subsets of X we have

$$\begin{aligned}
\bigcup_{j \in J} I_0(C_j) &= \bigcup_{j \in J} \bigcup_{c \in C_j} \{e \mid \exists n \in \mathbb{N}\colon e \in E_n \wedge \delta(e,c) < 1/n\} \\
&= \bigcup_{c \in \bigcup_{j \in J} C_j} \{e \mid \exists n \in \mathbb{N}\colon e \in E_n \wedge \delta(e,c) < 1/n\} \\
&= I_0\left(\bigcup_{j \in J} C_j\right)
\end{aligned}$$

Finally, the existence of a closed additive integral (specifically, $I_0^*(C) := \overline{I_0(C)}$) follows from Proposition 3.2. ■

References

[Hal84] Michael Hallet. *Cantorian Set Theory and Limitation of Size.* Clarendon Press, 1984.

[KM76] K. Kuratowski and A. Mostowski. *Set Theory.* North-Holland Publishing Company, 1976.